LINEAR SYSTEMS ON EDGE-WEIGHTED GRAPHS

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Abstract. Let R be any subring of the reals. We present a generalization of linear systems on graphs where divisors are R-valued functions on the set of vertices and graph edges are permitted to have nonegative weights in R. Using this generalization, we provide an independent proof of a Riemann-Roch formula, which implies the Riemann-Roch formula of Baker and Norine.

1. Introduction

Let R be any subring of the reals and G be a finite connected edge-weighted graph with vertex set $V = \{v_0, \ldots, v_n\}$ and nonnegative weight set $W = \{w_{ij} \mid i, j = 0, \ldots, n\}$ where each $w_{ij} \in R$. Multiple edges and loops are not allowed, and we set $w_{ij} = 0$ if v_i and v_j are not connected; otherwise, $w_{ij} > 0$. Note that $w_{ii} = 0$ and $w_{ij} = w_{ji}$. We will define the *degree* of a vertex v_j to be

$$\deg(v_j) = \sum_{i=0}^{n} w_{ij},$$

and the parameter g (the *genus* of the graph) to be

$$g = 1 + \sum_{i<j} w_{ij} - |V| = \sum_{i<j} w_{ij} - n.$$

Note that if $R = \mathbb{Z}$, these definitions coincide with the usual definitions for the vertex degree and genus of a multigraph where w_{ij} is the number of edges connecting vertices v_i and v_j.

A *divisor* on G is a function $D : V \to R$. The *degree* of a divisor D is defined as

$$\deg(D) = \sum_{v \in V} D(v).$$

For any $x \in \mathbb{R}$, we say that $D > x$ if $D(v) > x$ for each $v \in V$, and $D > D'$ if $D(v) > D'(v)$ for each $v \in V$.

The space of divisors on G, written $\mathrm{Div}(G)$, is a R-module and the subset of divisors on G with degree zero is denoted by $\mathrm{Div}^0(G)$.

The *canonical divisor* K is defined by $K(v) = \deg(v) - 2$ for any $v \in V$.

For any $j \in \{0, \ldots, n\}$, consider the divisor H_j, defined by

$$H_j(v_i) = \begin{cases} \deg(v_j) & \text{if } v_i = v_j \\ -w_{ij} & \text{otherwise.} \end{cases}$$

The *principal* divisors $\mathrm{PDiv}(G)$ are the $\mathbb{Z}$-linear combinations of the H_j divisors. Two divisors $D, D' \in \mathrm{Div}(G)$ are linearly equivalent, written $D \sim D'$, if and only if there is a $H \in \mathrm{PDiv}(G)$ such that $D - D' = H$.

For each divisor $D \in \mathrm{Div}(G)$, we associate a *complete linear system* $|D|$, which is defined as

$$|D| = \{D' \in \mathrm{Div}(G) \mid D' \sim D, D' > -1\}.$$

The *dimension* of $|D|$ is defined as

$$\ell(D) = \min_E \{\deg(E) \mid E \in \mathrm{Div}(G), E \geq 0, |D - E| = \emptyset\}.$$

We will show the following Riemann-Roch formula holds on G.

Theorem 1.1. *For any divisor $D \in \mathrm{Div}(G)$*

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g.$$

This theorem generalizes a similar statement for integral divisors on multigraphs proved by Baker and Norine in [1]. We showed in [5] that Theorem 1.1 follows from the result in [1]. Here, the proof of Theorem 1.1 is independent, relying on the following theorem.

Define the set of all divisors of degree $g - 1$ with empty linear systems by

$$\mathcal{N}(G) = \{D \in \mathrm{Div}(G) \mid \deg(D) = g - 1 \text{ and } |D| = \emptyset\}.$$

Theorem 1.2.

(1) *The set $\mathcal{N}(G)$ is symmetric with respect to K; that is, $N \in \mathcal{N}(G)$ if and only if $K - N \in \mathcal{N}(G)$.*
(2) *For any $D \in \mathrm{Div}(G)$, $|D| = \emptyset$ if and only if there is a $N \in \mathcal{N}(G)$ such that $D \leq N$.*

In the following section, we prove Theorem 1.1, assuming Theorem 1.2. The proof of Theorem 1.2, which relies on a normal form for divisors, up to linear equivalence, follows in the subsequent sections.

2. Proof of Riemann-Roch

For any $D \in \mathrm{Div}(G)$, define

$$\begin{aligned} D^+(v) &= \max(D(v), 0) \\ D^-(v) &= \min(D(v), 0). \end{aligned}$$

It follows directly from these definitions that for any $D \in \mathrm{Div}(G)$,

$$D = D^+ + D^-$$

and

$$\deg(D^+) = -\deg((-D)^-).$$

Lemma 2.1. *If statement (2) of Theorem 1.2 is true, then for any $D \in \mathrm{Div}(G)$,*

$$\ell(D) = \min_{N \in \mathcal{N}(G)} \deg((D - N)^+).$$

Proof. The definition of $\ell(D)$ is

$$\ell(D) = \min_E \{\deg(E) \mid E \geq 0, |D - E| = \emptyset.$$

Property (2) of Theorem 1.2 implies that

$$\begin{aligned} \ell(D) &= \min_{E,N} \{\deg(E) \mid E \geq 0, N \in \mathcal{N}(G), D - E \leq N\} \\ &= \min_{E,N} \{\deg(E) \mid E \geq 0, N \in \mathcal{N}(G), E \geq D - N\} \end{aligned}$$

or equivalently,

$$\ell(D) = \min_{N \in \mathcal{N}(G)} \deg((D-N)^+).$$

$\square$

Lemma 2.2. *If Theorem 1.2 holds, then for any $D \in \mathrm{Div}(G)$,*

$$\ell(K-D) = g - 1 - \deg(D) + \min_{M \in \mathcal{N}(G)} \deg((D-M)^+).$$

Proof. From property (2) of Theorem 1.2,

$$\begin{aligned}
\ell(K-D) &= \min_{E}\{\deg(E) \mid E \geq 0, |K-D-E| = \emptyset\} \\
&= \min_{E,M}\{\deg(E) \mid E \geq 0, M \in \mathcal{N}(G), K-D-E \leq M\}.
\end{aligned}$$

If $K-D-E \leq M$ for $M \in \mathcal{N}(G)$, then $D+E \geq K-M$. Property (1) of Theorem 1.2 implies that $K-M \in \mathcal{N}(G)$ if and only if $M \in \mathcal{N}(G)$, thus we have

$$\begin{aligned}
\ell(K-D) &= \min_{E,M}\{\deg(E) \mid E \geq 0, M \in \mathcal{N}(G), D+E \geq M\} \\
&= \min_{E,M}\{\deg(E) \mid E \geq 0, M \in \mathcal{N}(G), E \geq M-D\} \\
&= \min_{M \in \mathcal{N}(G)} \deg((M-D)^+).
\end{aligned}$$

Since $\deg((M-D)^+) = \deg(M-D) - \deg((M-D)^-)$, we have

$$\begin{aligned}
\ell(K-D) &= \min_{M \in \mathcal{N}(G)} \deg((M-D)^+) \\
&= \min_{M \in \mathcal{N}(G)} \big(\deg(M-D) - \deg((M-D)^-)\big) \\
&= \deg(M) - \deg(D) + \min_{M \in \mathcal{N}(G)} \big(-\deg((M-D)^-)\big) \\
&= g - 1 - \deg(D) + \min_{M \in \mathcal{N}(G)} \deg((D-M)^+).
\end{aligned}$$

$\square$

We now have the ingredients to prove Theorem 1.1.

Proof. (Theorem 1.1) Using Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\ell(D) - \ell(K-D) &= \left(\min_{N \in \mathcal{N}(G)} \deg((D-N)^+)\right) \\
&\quad - \left(g - 1 - \deg(D) + \min_{M \in \mathcal{N}(G)} \deg((D-M)^+)\right) \\
&= \deg(D) - g + 1 + \min_{N \in \mathcal{N}(G)} \deg((D-N)^+) \\
&\quad - \min_{M \in \mathcal{N}(G)} \deg((D-M)^+) \\
&= \deg(D) - g + 1.
\end{aligned}$$

$\square$

3. Reduced Divisors

Let $V_0 = V - \{v_0\}$. We say that a divisor $D \in \mathrm{Div}(G)$ is *reduced* if and only if

(1) $D(v) > -1$ for each $v \in V_0$, and
(2) for every $I \subset \{1, \dots, n\}$, there is a $v \in V_0$ such that

$$(D - \sum_{j \in I} H_j)(v) \leq -1.$$

Define $\mathcal{P}(G) \subset \mathrm{PDiv}(G)$ to be the set of non-negative, non-zero $\mathbb{Z}$-linear combinations of the H_j divisors for $j > 0$; that is, if $H \in \mathcal{P}(G)$ then there is a set of nonnegative integers $\{c_1, \dots, c_n\}$ such that

$$H = \sum_{j=1}^{n} c_j H_j.$$

Lemma 3.1. *Suppose a divisor $D(v) > -1$ for all $v \in V_0$, then D is reduced if and only if for every $H \in \mathcal{P}(G)$, there is a $v \in V_0$ such that*

$$(D - H)(v) \leq -1.$$

Proof. Assume that $D(v) > -1$ for all $v \in V_0$.

If for every $H \in \mathcal{P}(G)$, there is a $v \in V_0$ such that $(D - H)(v) \leq -1$, then D is clearly reduced, thus we need only show the converse is true.

Suppose that there exists a $H = \sum_{i=1}^{n} c_i H_i \in \mathcal{P}(G)$ such that

$$(D - H)(v) > -1$$

for all $v \in V_0$. This means that for each $j = 1, \dots, n$

$$D(v_j) > c_j \deg(v_j) - \sum_{i=1}^{n} c_i w_{ij} - 1.$$

Let $\alpha = \max\{c_1, \dots, c_n\}$ and for each $i = 1, \dots, n$ set

$$b_i = \begin{cases} 1 & \text{if } c_i = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for each $j \in \{1, \dots, n\}$

$$D(v_j) > b_j \deg(v_j) - \sum_{i=1}^{n} b_i w_{ij} - 1.$$

If $b_j = 0$, we have

$$b_j \deg(v_j) - \sum_{i=1}^{n} b_i w_{ij} - 1 = -\sum_{i=1}^{n} b_i w_{ij} - 1 \leq -1 < D(v_j).$$

Define the index sets A_j and B_j as

$$\begin{aligned} A_j &= \{i > 0 \mid w_{ij} > 0 \text{ and } c_i < \alpha\} \\ B_j &= \{i > 0 \mid w_{ij} > 0 \text{ and } c_i = \alpha\} \end{aligned}$$

and note that

$$\sum_{i=1}^{n} c_i w_{ij} = \sum_{i \in A_j} c_i w_{ij} + \alpha \sum_{i \in B_j} w_{ij}.$$

If $b_j = 1$, then $c_j = \alpha$ and

$$\begin{aligned} D(v_j) &> c_j \deg(v_j) - \sum_{i=1}^{n} c_i w_{ij} - 1 \\ &= \alpha \deg(v_j) - \sum_{i \in A_j} c_i w_{ij} - \alpha \sum_{i \in B_j} w_{ij} - 1 \\ &= \alpha(w_{0j} + \sum_{i \in A_j} w_{ij} + \sum_{i \in B_j} w_{ij}) - \sum_{i \in A_j} c_i w_{ij} - \alpha \sum_{i \in B_j} w_{ij} - 1 \\ &= \alpha w_{0j} + \sum_{i \in A_j} (\alpha - c_i) w_{ij} - 1 \\ &\geq w_{0j} + \sum_{i \in A_j} w_{ij} - 1. \end{aligned}$$

Also, we have

$$\begin{aligned} b_j \deg(v_j) - \sum_{i=1}^{n} b_i w_{ij} &= \deg(v_j) - \sum_{i \in B_j} w_{ij} \\ &= w_{0j} + \sum_{i \in A_j} w_{ij} + \sum_{i \in B_j} w_{ij} - \sum_{i \in B_j} w_{ij} \\ &= w_{0j} + \sum_{i \in A_j} w_{ij} \end{aligned}$$

thus $D(v_j) > b_j \deg(v_j) - \sum_{i=1}^{n} b_i w_{ij} - 1$ for each $j = 1, \ldots, n$, hence D is not reduced. $\square$

Let Δ_0 be the *edge-weighted reduced Laplacian* of G, which can be represented by the $n \times n$ matrix

$$\Delta_0 = \begin{pmatrix} \deg(v_1) & -w_{12} & \cdots & -w_{1n} \\ -w_{12} & \deg(v_2) & \cdots & -w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{1n} & -w_{2n} & \cdots & \deg(v_n) \end{pmatrix}.$$

For $x = (x_1, \ldots, x_n)$ and $y = (y_i, \ldots, y_n)$, we say that $x > y$ if and only if $x_i > y_i$ for each i; for any scalar $a \in \mathbb{R}$, $x > a$ if and only if $x_i > a$ for each i; finally, we define

$$\max(x, y) = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$$

and $x > a$ if and only if $x_i > a$ for each i; finally, we define

$$\min(x, y) = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}).$$

We showed in [5] that Δ_0 is monotone; that is, for any $x \in \mathbb{R}^n$, if $\Delta_0 x \geq 0$, then $x \geq 0$. Monotonicity implies that Δ_0^{-1} exists and is nonnegative, and that if $x, y \geq 0$ with $y \geq \Delta_0 x$, then $\Delta_0^{-1} y \geq x$ (see [2]).

Lemma 3.2. *For any $z \in \mathbb{R}^n$ such that $z \geq 0$, there is a $c \in \mathbb{Z}^n$ such that $c \geq 0$ and $\Delta_0 c \geq z$.*

Proof. Fix $z \in \mathbb{R}^n$ with $z \geq 0$. Let $C_0 = \{x \in \mathbb{R}^n \mid \Delta_0 x \geq 0\}$, $C_z = \{x \in \mathbb{R}^n \mid \Delta_0 x \geq z\}$ and $K = \{x \in \mathbb{R}^n \mid x \geq 0\}$. Since Δ_0 is monotone, $C_z \neq \emptyset$ and $C_z \subset C_0 \subset K$.

Let $x, y \in C_0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta \geq 0$, then

$$\Delta_0(\alpha x + \beta y) = \alpha \Delta_0 x + \beta \Delta_0 y \geq 0,$$

and $\alpha x + \beta y \in C_0$. Thus C_0 is a convex cone, and since Δ_0 is injective, C_0 has an interior. Let $v = \Delta_0^{-1} z$. For any $x \in C_z$,

$$\Delta_0(x - v) = \Delta_0 x - z \geq 0$$

so $x - v \in C_0$ and $C_z - v$ is also a convex cone, thus C_z is a convex affine cone with an interior. Hence $C_z \cap \mathbb{Z}^n \neq \emptyset$. $\square$

Define the function $\phi : \mathrm{Div}(G) \to R^n$ as

$$\phi(D) = (D(v_1), \ldots, D(v_n)).$$

We can represent any $H = \sum_{i=1}^n c_i H_i \in \mathrm{PDiv}(G)$ as

$$\phi(H) = \Delta_0 c$$

where $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$; $H(v_0)$ can be recovered by

$$H(v_0) = \sum_{i=1}^n c_i H_i(v_0) = -\sum_{i=1}^n c_i w_{0i}.$$

For any $d \in R^n$, define $\mathcal{A}(d) \subset \mathbb{Z}^n$ be

$$\mathcal{A}(d) = \{c \in \mathbb{Z}^n \mid c \geq 0, d - \Delta_0 c > 0\}.$$

Note that if $d > 0$, $\mathcal{A}(d) \neq \emptyset$ since the zero vector $(0, \ldots, 0) \in \mathcal{A}(d)$.

Let $D \in \mathrm{Div}(G)$ and again set $d = \phi(D+1)$. Using the above notation, it follows directly from Lemma 3.1 that D is reduced if and only if

(1) $d > 0$, and
(2) $\mathcal{A}(d) = \{0\}$, the zero vector in $\mathbb{Z}^n$.

Lemma 3.3. *Let $D \in \mathrm{Div}(G)$ and set $d = \phi(D+1)$. If $d > 0$ then $\mathcal{A}(d)$ is bounded; that is, there exists $b \in \mathbb{R}^n$ such that $b \geq c$ for all $c \in \mathcal{A}(d)$.*

Proof. Suppose $d > 0$ and $c \in \mathcal{A}(d)$. Since $d > \Delta_0 c$, since Δ_0 is monotone, we have $b = \Delta_0^{-1} d \geq c$. $\square$

Lemma 3.4. *Let $D \in \mathrm{Div}(G)$ and set $d = \phi(D+1)$. If $d > 0$ and $c, c' \in \mathcal{A}(d)$, then $\max(c, c') \in \mathcal{A}(d)$.*

Proof. Suppose $d > 0$ and $c, c' \in \mathcal{A}(d)$, then we have both $d - \Delta_0 c > 0$ and $d - \Delta_0 c' > 0$. We can write the jth component of $\Delta_0 c$ as

$$\begin{aligned}
(\Delta_0 c)_j &= c_j \deg(v_j) - \sum_{i=1}^n c_i w_{ij} \\
&= \sum_{i=0}^n c_j w_{ij} - \sum_{i=1}^n c_i w_{ij} \\
&= c_j w_{0j} + \sum_{i=1}^n (c_j - c_i) w_{ij}
\end{aligned}$$

and similarly for $\Delta_0 c'$. For $d = (d_1, \ldots, d_n)$, we then have

$$\begin{aligned} d_j &> c_j w_{0j} + \sum_{i=1}^{n} (c_j - c_i) w_{ij} \\ d_j &> c'_j w_{0j} + \sum_{i=1}^{n} (c'_j - c'_i) w_{ij} \end{aligned}$$

for each j. If $\max\{c_j, c'_j\} = c_j$, then

$$d_j > c_j w_{0j} + \sum_{i=1}^{n} (c_j - \max\{c_i, c'_i\}) w_{ij}$$

and if $\max\{c_j, c'_j\} = c'_j$,

$$d_j > c'_j w_{0j} + \sum_{i=1}^{n} (c'_j - \max\{c_i, c'_i\}) w_{ij}.$$

We can combine these two relations to get

$$d_j > \max\{c_j, c'_j\} w_{0j} + \sum_{i=1}^{n} (\max\{c_j, c'_j\} - \max\{c_i, c'_i\}) w_{ij}$$

for each $j > 0$, thus

$$d - \Delta_0 \max(c, c') > 0$$

and $\max(c, c') \in \mathcal{A}(d)$. □

Theorem 3.5. *For any $D \in \mathrm{Div}(G)$ there is a unique $D_0 \sim D$ such that D_0 is reduced.*

Proof. Let $D \in \mathrm{Div}(G)$. By Lemma 3.2, choose $c \in \mathbb{Z}^n$ so that $\Delta_0 c \geq \phi(-(D^-))$, noting that $-(D^-) \geq 0$. Set $d = \phi(D + 1) + \Delta_0 c$, which guarantees that $d > 0$.

By Lemmas 3.3 and 3.4, $\hat{c} = \max\{c' \mid c' \in \mathcal{A}(d)\}$ exists and is unique. We claim that $\mathcal{A}(d - \Delta_0 \hat{c}) = \{0\}$. Let $c' \in \mathcal{A}(d - \Delta_0 \hat{c})$, then

$$d - \Delta_0 \hat{c} - \Delta_0 c' = d - \Delta_0 (\hat{c} + c') > 0,$$

thus $\hat{c} + c' \in \mathcal{A}(d)$. Since $\hat{c}$ is maximal in $\mathcal{A}(d)$ and $\hat{c}, c' \geq 0$, we must have $c' = 0$. It then follows from Lemma 3.1 that there is a unique reduced divisor D_0 such that $\phi(D_0) = d - \Delta_0 \hat{c}$, where $D_0 \sim D$ since the translations $\Delta_0 c$ and $\Delta_0 \hat{c}$ correspond to some $H \in \mathrm{PDiv}(G)$. □

4. Empty Linear Systems

In this section, we will exploit properties of reduced divisors to determine the set of divisors which have empty linear systems. We begin with the following property of reduced divisors.

Lemma 4.1. *If $D \in \mathrm{Div}(G)$ is reduced, then*

$$\sum_{v \in V_0} D(v) \leq g,$$

with equality if and only if there exists a permutation $(j_1, j_2, \ldots, j_n)$ *of* $(1, 2, \ldots, n)$ *such that*

$$D(v_{j_k}) = \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

for each $k = 1, \ldots, n$, *where* $j_0 = 0$.

Proof. Suppose that D is reduced, then for every $I \subset \{1, \ldots, n\}$, there is a $j \in V_0$ such that

$$D(v_j) \leq \deg(v_j) - \sum_{i \in I} w_{ij} - 1 = \sum_{i \notin I} w_{ij} - 1. \tag{4.1}$$

Suppose that $I = I_0 = \{1, \ldots, n\}$, and that that (4.1) is satified for $j = j_1 \in I_0$, then

$$D(v_{j_1}) \leq \sum_{i \notin I_0} w_{ij_1} - 1 = w_{0j_1} - 1.$$

Now let $I = I_1 = I_0 - \{j_1\}$, then (4.1) is satified for $j = j_2 \in I_1$ so that

$$D(v_{j_2}) \leq \sum_{i \notin I_1} w_{ij_2} - 1 = w_{0j_2} + w_{j_1 j_2} - 1.$$

Similarly, for $I = I_2 = I_1 - \{j_2\}$, (4.1) is satisfied for $j = j_3 \in I_2$ and

$$D(v_{j_3}) \leq \sum_{i \notin I_2} w_{ij_3} - 1 = w_{0j_3} + w_{j_1 j_3} + w_{j_2 j_3} - 1.$$

Continuing this process, let $I_k = I_{k-1} - \{j_k\}$ for $k = 1, \ldots, n-1$, where $j = j_k \in I_{k-1}$ satisfies (4.1) for I_{k-1}, and we have in general

$$D(v_{j_k}) \leq \sum_{i=0}^{k-1} w_{j_i j_k} - 1 \tag{4.2}$$

where $j_0 = 0$. Note that the resulting n-tuple $(j_1, j_2, \ldots, j_n)$ is a permutation of $(1, 2, \ldots, n)$. If we rewrite (4.2) as

$$D(v_{j_k}) - \sum_{i=0}^{k-1} w_{j_i j_k} + 1 \leq 0$$

and sum over all k, we have

$$\sum_{k=1}^{n} \left(D(v_{j_k}) - \sum_{i=0}^{k-1} w_{j_i j_k} + 1 \right) = \sum_{j=1}^{n} D(v_j) - \sum_{i<j} w_{ij} + n \leq 0$$

or equivalently,

$$\sum_{v \in V_0} D(v) \leq \sum_{i<j} w_{ij} - n = g.$$

For the equality condition, we assume again the D is reduced and first note that if

$$D(v_{j_k}) = \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

holds for some $(j_1, \ldots, j_n)$ for each $k = 1, \ldots, n$, then

$$\sum_{v \in V_0} D(v) = g$$

follows directly since

$$\sum_{i<j} w_{ij} - n = g.$$

For the other direction, if

$$\sum_{v \in V_0} D(v) = g,$$

since D is reduced, (4.2) holds at each k for some permutation $(j_1, \dots, j_n)$, thus the only way that we can have equality is for

$$D(v_{j_k}) = \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

for each $k = 1, \dots, n$. $\square$

An immediate application of Lemma 4.1 and Theorem 3.5 gives a sufficient condition for a divisor to have a nonempty linear system.

Lemma 4.2. *Let $D \in \mathrm{Div}(G)$. If $\deg(D) > g - 1$ then $|D| \neq \emptyset$.*

Proof. Let D be a divisor with $\deg(D) > g - 1$, and let D_0 be the unique reduced divisor such that $D_0 \sim D$ from Theorem 3.5. By Lemma 4.1

$$\sum_{v \in V_0} D_0(v) \leq g.$$

By assumption we have

$$\deg(D) = \deg(D_0) = D_0(v_0) + \sum_{v \in V_0} D_0(v) > g - 1,$$

or equivalently

$$D_0(v_0) > -\sum_{v \in V_0} D_0(v) + g - 1,$$

thus $D_0(v_0) > -1$. Since $D_0(v) > -1$ for each $v \in V_0$, $|D| \neq \emptyset$. $\square$

Lemma 4.3. *If D_0 be a reduced divisor, then $|D_0| \neq \emptyset$ if and only if $D_0(v_0) > -1$.*

Proof. Let $D_0 \in \mathrm{Div}(G)$ be reduced. If $D_0(v_0) > -1$, then $D_0(v) > -1$ for all $v \in V$ and $|D_0| \neq \emptyset$.

Now assume that $|D_0| \neq \emptyset$, thus there is a $P \in \mathrm{PDiv}(G)$ such that $D_0 + P > -1$. Since D_0 is reduced, the only $P \in \mathrm{PDiv}(G)$ which would satisfy $D_0 + P > -1$ must have $P(v) \geq 0$ for all $v \in V_0$. Since $\deg(P) = 0$, $P(v_0) \leq 0$, thus we must have $D_0 > -1$ in order for $|D_0|$ to be nonempty. $\square$

Lemma 4.4. *If D_0 is a reduced divisor with $\deg(D_0) = g - 1$ and $|D_0| = \emptyset$, then*

$$D_0(v_{j_l}) = \begin{cases} -1 & l = 0 \\ \sum_{i=0}^{l-1} w_{j_i j_l} - 1 & l > 0 \end{cases}$$

where $j_0 = 0$ and $(j_1, \dots, j_n)$ is a permutation of $(1, \dots, n)$.

Proof. By Lemma 4.3 $D_0(v_0) \leq -1$, thus by Lemma 4.1 we then have

$$\sum_{i=1}^{n} D_0(v_i) = g,$$

$D_0(v_0) = -1$, and

$$D_0(v_{j_l}) = \sum_{i=0}^{l-1} w_{j_i j_l} - 1$$

for some permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$. $\square$

We will denote the reduced divisors in Lemma 4.4 as

$$\mathcal{N}_0(G) = \{D \in \mathrm{Div}(G) \mid |D| = \emptyset, \deg(D) = g - 1, D \text{ is reduced } \} \subset \mathcal{N}(G),$$

noting that $|\mathcal{N}_0(G)| \leq n!$.

A direct consequence of Lemma 4.4 then gives us the composition of $\mathcal{N}(G)$, which is a lattice generated by $\mathcal{N}_0(G)$.

Lemma 4.5. $\mathcal{N}(G) = \{D \in \mathrm{Div}(G) \mid D \sim D_0$ *where* $D_0 \in \mathcal{N}_0(G)\}$.

Proof. If $D \in \mathcal{N}(G)$, then by Lemma 3.5 there is a $D_0 \in \mathcal{N}_0(G)$ that is linearly equivalent to D. $\square$

We can now prove Theorem 1.2.

Proof. (Theorem 1.2)

(1) Since any $D \in \mathcal{N}(G)$ can be written as $D = N_0 + P$ for some $P \in \mathrm{PDiv}(G)$ and $N_0 \in \mathcal{N}_0(G)$, it is sufficient to assume $D = N_0$.

By Lemma 4.4,

$$N_0(v_{j_0}) = -1$$

and

$$N_0(v_{j_k}) = \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

for some permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$ with $j_0 = 0$. Since

$$K(v_i) = \sum_{j=0}^{n} w_{ij} - 2,$$

for $k > 0$ we have

$$(K - D)(v_{j_k}) = \sum_{i=0}^{n} w_{j_i j_k} - 2 - \sum_{i=0}^{k-1} w_{j_i j_k} + 1 = \sum_{i=k}^{n} w_{j_i j_k} - 1$$

and for $k = 0$

$$(K - D)(v_{j_0}) = \sum_{i=0}^{n} w_{j_i j_0} - 1 = \deg(v_{j_0}) - 1.$$

If we subtract $H_0 \in \mathrm{PDiv}(G)$ from $K - D$, we have

$$(K - D - H_0)(v_{j_0}) = -1$$

and for $k > 0$,

$$(K - D - H_0)(v_{j_k}) = \sum_{i=k}^{n} w_{j_i j_k} - 1 + w_{j_0 j_k}.$$

Let $l_0 = j_0 = 0$ and $l_k = j_{n-k+1}$ for $k = 1, \ldots, n$; then $(l_1, \ldots, l_n)$ is permutation of $(1, \ldots, n)$ and

$$(K - D - H_0)(v_{l_k}) = \sum_{i=0}^{k-1} w_{l_i l_k} - 1,$$

thus $K - D - H_0 \in \mathcal{N}_0(G)$ and $K - D \in \mathcal{N}(G)$.

Now assume that $K - D \in \mathcal{N}_0(G)$. Let $D' = K - D$, and from above we have $K - D' = D \in \mathcal{N}(G)$.

(2) Let $D \in \mathrm{Div}(G)$ with $|D| = \emptyset$. By Lemma 3.5, there is a unique reduced divisor $D_0 \sim D$. Since $|D_0| = \emptyset$, Lemma 4.3 implies that $D_0(v_0) \leq -1$. By the proof of Lemma 4.1, we have that (4.2) holds for each $D_0(v)$ where $v \in V_0$, so for some permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$,

$$D_0(v_{j_k}) \leq \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

and thus $D_0 \leq N_0$ for one of the $N_0 \in \mathcal{N}_0(G)$. Let $P \in \mathrm{PDiv}(G)$ such that $D = D_0 + P$, and let $N = N_0 + P$. Then we have $D \leq N$ where $N \in \mathcal{N}(G)$. $\square$

References

[1] Matthew Baker and Serguei Norine. Riemann-Roch and Abel-Jacobi Theory on a Finite Graph. *Advances in Mathematics* 215 (2007).

[2] Bernman, Abraham and Plemmons, Robert J., *Nonnegative Matrices in the Mathematical Sciences*, Classics in Applied Mathematics, SIAM, Philadelphia, PA, 1994.

[3] N.L. Biggs. Chip-Firing and the Critical Group of a Graph. *Journal of Algebraic Combinatorics* 9 (1999).

[4] Andreas Gathmann and Michael Kerber. A Riemann-Roch Theorem in Tropical Geometry. *Mathematische Zeitschrift* 259 (2008).

[5] Rodney James and Rick Miranda. A Riemann-Roch theorem for edge-weighted graphs. arXiv:0908.1197 (2009).

[6] A. Postnikov and B. Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. *Transactions of the American Mathematical Society* 356 (8) (2004).

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